JOURNAL OF COMBINATORIAL THEORY (A) 16, 51-56 (1974)

Unitary Block Designs*

D. E. TAYLOR

Department of Mathematics, La Trobe University, Bundoora, Victoria, Australia 3083

Communicated by Marshall Hall, Jr.

Received May 10, 1972

Given a projective plane $\mathfrak{C}$ over the field of q^2 elements and a unitary polarity π of $\mathfrak{C}$ it is possible to construct the well-known unitary design $\mathfrak{U}$ whose points are the absolute points of π and whose blocks are the non-absolute lines of π. A relation of perpendicularity is defined between blocks and it is shown that this relation can be described in terms of the incidence structure of $\mathfrak{U}$. The projective plane $\mathfrak{C}$ together with the polarity π can then be reconstructed from the design $\mathfrak{U}$ in such a way that any automorphism of $\mathfrak{U}$ extends to a collineation of $\mathfrak{C}$ which commutes with π.

1. INTRODUCTION

Recently, M. O'Nan [4] considered the unitary design associated with the three-dimensional unitary group over a finite field of q^2 elements and showed that 1st automorphism group is $P\Gamma L(3, q^2)$. This note uses ideas of O'Nan [4] and Dembowski and Hughes [2] to show how the projective plane together with its unitary polarity may be reconstructed directly from the unitary design. The main result of O'Nan [4] is obtained as a corollary.

The notation and results of Dembowski [1, §1.4] will be freely used.

2. THE UNITARY DESIGN

This section contains the basic facts about three-dimensional unitary geometry that will be needed later. More details are to be found in Dembowski [I, §1.4] or Lüneburg [3, §10].

Let K be the field $GF(q^2)$, where q is the power of a prime and let K_1 = GF(q) be the fixed field of the automorphism $x \to \bar{x} = x^q$ of K.

* This research was partially supported by a CSIRO Overseas Studentship at the Mathematical Institute, 24-29 St. Giles, Oxford.

Let V be a three-dimensional vector space over K and let s be a non-degenerate sesquilinear form on V such that $s(x, y) = \overline{s(y, x)}$ for all $x, y \in$ v.

Given a subspace U of V we define the polar subspace of U by

$$U^\pi = \{x \in V \quad s(x, U) = 0\}.$$

The correspondence $\pi: U \to U^\pi$ is a unitary polarity of the projective plane $\mathfrak{E} = \mathfrak{P}(V)$. A point p of $\mathfrak{E}$ is ***absolute*** if p is incident with its polar line p^π. Similarly, a line L of $\mathfrak{E}$ is ***absolute*** if L is incident with its polar point L^π. The set of absolute points of $\mathfrak{E}$ will be denoted by Ω. It follows from Dembowski [1, 1.4.47] that

$$|\Omega| = q^3 + 1.$$

The group of collineations of $\mathfrak{E}$ leaving Ω invariant is the group $\mathrm{P\Gamma L}(3, q^2)$ of those collineations of $\mathfrak{E}$ which commute with π (Dembowski [1, p. 44]).

Let us now choose two points of Ω and denote them by the symbols co and 0. If L is the line determined by 0 and ∞ we may choose vectors e_1, e_2, and e_3 in V which represent the points 0, L^π and ∞, respectively, and which satisfy the following:

$$s(e_1, e_1) = s(e_3, e_3) = s(e_1, e_2) = s(e_2, e_3) = 0,$$
$$s(e_2, e_2) = s(e_1, e_3) = 1.$$

Moreover, e_1, e_2, and e_3 form a basis for V and, if $p \in \mathfrak{E}$ has homogeneous coordinates (x_1, x_2, x_3) with respect to this basis, then $p \in \Omega$ if and only if $x_1\bar{x}_3 + x_2\bar{x}_2 + x_3\bar{x}_1 = 0$. Since co and 0 were chosen arbitrarily in Ω it also follows that $\mathrm{P\Gamma U}(3, q^2)$ acts doubly transitively on Ω.

Let L_∞ be the line ∞^π and let $\mathfrak{A}$ be the affine plane obtained by deleting L_∞ from $\mathfrak{E}$. If $p \in \mathfrak{A}$, then $x_1 \neq 0$ and p has affine coordinates (x, y), where $x = x_2x_1^{-1}$ and $y = x_3x_1^{-1}$. In the next section, it will be convenient to refer to the lines of $\mathfrak{E}$ by their equations in $\mathfrak{A}$. If N and Tr are, respectively, the norm and trace from K to K_1, then

$$\Omega = \{\infty\} \cup \{(x, y) \in \mathfrak{A} \quad \mathrm{N}(x) + \mathrm{Tr}(y) = 0\}.$$

It follows easily from this description that any line of $\mathfrak{E}$ which contains ∞ is either the absolute line L_∞ or is a non-absolute line which is incident with $q + 1$ points of Ω. Since ∞ can be any point of Ω there are $(q^3 + 1)\, q^2/(q + 1) = q^2(q^2 - q + 1)$ lines which meet Ω in $q + 1$ points.

These lines together with the $q^3 + 1$ absolute lines (which meet Ω in just one point) account for all the lines of $\mathfrak{E}$. We may therefore form a design $\mathfrak{U}$ whose point set is Ω, whose set $\mathfrak{B}$ of blocks is the set of non-absolute lines, and whose incidence relation is the restriction of that of $\mathfrak{E}$ (Dembowski [1, p. 54]). This is the unitary design associated with $\mathfrak{E}$ and π. Its parameters are

$$v = q^3 + 1, \quad b = q^2(q^2 - q + 1), \quad r = q^2, \quad k = q + 1, \quad \lambda = 1.$$

If L and M are non-absolute lines we say that L and M are ***perpendicular*** and write $L \perp M$ whenever L^π is incident with M (equivalently, whenever M^π is incident with L) (Dembowski [1, p. 155]).

3. The Construction of $(\mathfrak{E}, \pi)$ from $\mathfrak{U}$

The main task of this section is to show that $\perp$ is determined by the incidence relation of $\mathfrak{U}$. It is then easy to reconstruct $\mathfrak{E}$ and π from $\mathfrak{U}$ and $\perp$. We first establish two lemmas:

LEMMA 3.1. ***Any line L of $\mathfrak{B}$ is perpendicular to $q^2 - q$ other lines of $\mathfrak{B}$ and every point of Ω is incident with just one of these $q^2 - q + 1$ lines.***

Proof. Let us set $p = L^\pi$ and let $p_1, ..., p_{q+1}$ be the points of Ω incident with L. Then $p \notin \Omega$ and p is incident with p_i^π for $1 \leqslant i \leqslant q + 1$. Since the lines p_i^π are absolute, the non-absolute lines perpendicular to L are the $(q^2 + 1) - (q + 1)$ lines incident with p which meet L outside Ω. If p' is any point of Ω, then either p' is incident with L or the line determined by p and p' is perpendicular to L. Since each line is incident with $q + 1$ points of Ω, it follows that each point of Ω not incident with L is incident with a unique line perpendicular to L.

LEMMA 3.2 (O'Nan [4]). ***For $a \in K$, $\lambda \in K_1$, and $\lambda \neq 0$, the set $C = \{z \mid N(z - a) = \lambda\}$ has $q + 1$ elements and $a = \sum_{z \in C} z$.***

Proof. Since $|\operatorname{Ker} N| = q + 1$, C has $q + 1$ elements. Now choose $\mu \in N$ such that $N(\mu) = \lambda^{-1}$ and put

$$D = \mu C - a\mu = \{\zeta \mid N(\zeta) = 1\}.$$

Thus D is the set of all $(q + 1)$-th roots of unity and therefore $\sum_{\zeta \in D} \zeta = 0$. The lemma now follows from the equation

$$0 = \sum_{z \in C} \mu(z - a) = \mu \Big(\sum_{z \in C} z - (q + 1)\, a\Big).$$

THEOREM 3.3. *Let L be a non-absolute line and let $\mathfrak{M}$ be a set of $q^2 - q$ non-absolute lines such that any point of Ω not incident with L is incident with just one line of $\mathfrak{M}$. Then $\mathfrak{M}$ is the set of lines perpendicular to L if and only if for each point p of Ω incident with L the set $\mathfrak{N}^p$ of non-absolute lines $\neq L$ which are incident with p can be partitioned into $q - 1$ subsets $\mathfrak{N}_1{}^p,..., \mathfrak{N}_{q-1}^p$ and $\mathfrak{M}$ can be partitioned into $q - 1$ subsets $\mathfrak{M}_1{}^p,..., \mathfrak{M}_{q-1}^p$ such that the following two conditions hold:*

(i) $\mathfrak{M}_i{}^p = q$ *and* $\mathfrak{N}_i{}^p = q + 1$, *for* $1 \leqslant i \leqslant q - 1$.

(ii) *If $M \in \mathfrak{M}_i{}^p$ and $N \in \mathfrak{N}_j{}^p$, then M meets N in a point of Ω if and only if $i = j$.*

Proof. Using the results of the previous section, we may choose the notation so that p = co and L is incident with co and 0. Thus L corresponds to the line in $\mathfrak{A}$ with equation x = 0 and L^π has representative e_2 in the vector space V. Henceforth we shall refer to the lines of $\mathfrak{E}$ by their equations in $\mathfrak{A}$. It follows that the lines perpendicular to L have equations y = b, where Tr(b) $\neq$ 0 and the lines of $\mathfrak{N}^\infty$ have equations x = a, where $a \neq 0$.

Let us suppose the field K_1 to be $\{0, \lambda_1 ,..., \lambda_{q-1}\}$ and let $\mathfrak{L}$ denote the set of lines perpendicular to L. If we define $\mathfrak{L}_i{}^\infty$ to be the subset of $\mathfrak{L}$ whose lines have equations y = b, where Tr(b) = $-\lambda_i$ and $\mathfrak{N}_i{}^\infty$ to be the subset of $\mathfrak{N}^\infty$ whose lines have equations x = a where $N(a) = \lambda_i$, for $1 \leqslant i \leqslant q - 1$, then the conditions of the theorem are satisfied with $\mathfrak{L}$ in place of $\mathfrak{M}$.

Now suppose that $\mathfrak{M}$ is a set of lines satisfying the conditions of the theorem. Since L has equation x = 0, the lines of $\mathfrak{M}$ must have equations $y = a_i x + b_i$, where $a_i, b_i \in K$, for $1 \leqslant i \leqslant q^2 - q$, and we may suppose the lines of $\mathfrak{M}_1{}^\infty$ to be $y = a_i x + b_i$, for $1 \leqslant i \leqslant q$. The lines of $\mathfrak{N}_1{}^\infty$ are $x = c_i$, for $1 \leqslant i \leqslant q + 1$ and it follows from condition (ii) that the elements $c_1 ,..., c_{q+1}$ all satisfy the q equations

$$N(x) + \mathrm{Tr}(a_i x + b) = 0, \qquad 1 \leqslant i \leqslant q.$$

Since these equations may be written in the form

$$N(x + \bar{a}_i) = N(a_i) - \mathrm{Tr}(b_i), \qquad 1 \leqslant i \leqslant q,$$

it follows from Lemma 3.2 that we have $a, = \cdots = a, = a$, say, and $b_1 + \bar{b}_1 = \cdots = b_q = \mu$, say.

From $\mathfrak{M}_1{}^\infty = q$ it follows that for some j we have $\mathfrak{M}_1{}^\infty \cap \mathfrak{M}_j{}^0 > 1$ and we may therefore suppose that $y = ax + b_1$ and $y = ax + b_2$ are lines of $\mathfrak{M}_1{}^\infty \cap \mathfrak{M}_j{}^0$, where $b_1 \neq b_2$. Now the lines of $\mathfrak{N}_j{}^0$ have equations $y = d_i x$, where $d_i \neq 0$, for $1 \leqslant i \leqslant q + 1$. The line $y = d_i x$ meets the line $y = ax + b_k$, $k = 1, 2$ in the point $(b_k(d_i - a)^{-1}, ab_k(d_i - a)^{-1} + b_k)$, provided $d_k \neq a$. Since each line of $\mathfrak{M}_j^0$ meets each line of $\mathfrak{N}_j^0$ in a point of Ω it follows that the elements $e_i = (d_i \quad a)^{-1}$, $1 \leqslant i \leqslant q + 1$ are all solutions of the equations

$$\mathrm{N}(x + \bar{a}b_i^{-1}) = \mathrm{N}(ab_k^{-1}) - \mathrm{Tr}(b_k^{-1}),$$

where k is 1 or 2.

By Lemma 3.2 we obtain $\bar{a}b_1^{-1} = \bar{a}b_2^{-1}$ and since $b_1 \neq b_2$ we must have a = 0. Moreover, $\mathfrak{M}_1^\infty$ was an arbitrary element of the partition so we have shown the lines in $\mathfrak{M}$ have equations $y = b$, where $\mathrm{Tr}(b) \neq 0$. But these are just the lines perpendicular to L and the theorem is proved.

COROLLARY 3.4. *Any automorphism of the design $\mathfrak{U}$ preserves $\perp$.*

Having shown that the relation of perpendicularity in $\mathfrak{U}$ can be defined by means of incidence we go on to construct a projective plane from $\mathfrak{U}$ whose relation of incidence depends only on incidence in $\mathfrak{U}$.

CONSTRUCTION 3.5. For each point $p \in \Omega$ we introduce a new symbol $[p]$ and for each line $L \in \mathfrak{B}$ we introduce a new symbol $[L]$. We then define

$$\Pi = \Omega \cup \{[L] \mid L \in \mathfrak{B}\} \qquad \text{and} \qquad \Lambda = \mathfrak{B} \cup \{[p] \mid p \in \Omega\}.$$

An incidence relation $I \subseteq \Pi \times \Lambda$ is defined by

$$\begin{aligned} &pI[q], && \text{if } p = q, \\ &pIL, && \text{if } p \text{ is incident with } L \text{ in } \mathfrak{U}, \\ &[L]\,I[p], && \text{if } p \text{ is incident with } L \text{ in } \mathfrak{U}, \\ &[L]\,IM, && \text{if } L \perp M, \end{aligned}$$

where $p, q \in \Omega$, and $L, M \in 23$.

If ϕ is an automorphism of $\mathfrak{U}$, we can extend ϕ to the incidence structure $\mathfrak{F} = (\Pi, \Lambda, I)$ by defining

$$[p]\phi = [p\phi] \qquad \text{and} \qquad [L]\phi = [L\phi],$$

where $p \in \Omega$ and $L \in \mathfrak{B}$. It follows from the previous theorem that ϕ extends to an automorphism of $\mathfrak{F}$. We finally define a bijection $\theta: \mathfrak{F} \to \mathfrak{E}$ as follows:

$$\begin{aligned} p\theta &= p \qquad \text{and} \qquad [L]\theta = L, \\ L\theta &= L \qquad \text{and} \qquad [p]\theta = P. \end{aligned}$$

THEOREM 3.6. ***The map θ: $\mathfrak{F} \to \mathfrak{E}$ is an isomorphism ofprojective planes. Identifying $\mathfrak{F}$ with $\mathfrak{E}$ by means of this isomorphism, any automorphism of $\mathfrak{U}$ is the restriction of a collineation of $\mathfrak{E}$ which commutes with the polarity π.***

Proof. The map θ is easily verified to be an isomorphism of incidence structures, hence $\mathfrak{F}$ is a projective plane. The polarity p of $\mathfrak{F}$ which corresponds to π under this isomorphism is given by $p^\rho = [p]$, $[p]^\rho = p$, $L^\rho = [L]$ and $[L]^\rho = L$. Thus any automorphism of $\mathfrak{U}$ extends to a collineation of $\mathfrak{F}$ which commutes with p and hence any automorphism of $\mathfrak{U}$ is the restriction of a collineation of $\mathfrak{F}$ which commutes with p. Identifying $\mathfrak{F}$ with $\mathfrak{E}$ we obtain the theorem. The main result of O'Nan [4] is now obtained as

COROLLARY ***3.1. The automorphism group of $\mathfrak{U}$ is*** $\mathrm{P\Gamma L}(3, q^2)$.

Remark. In $\mathrm{P\Gamma L}(3, q^2)$ the stabilizer of a line $L \in \mathfrak{B}$ is the centralizer of the subgroup W_L which fixes L pointwise. It is not hard to see that $L \perp$ A4 if and only if $[W_L, W_M] = 1$. Moreover, when q is odd, W_L contains a unique involution i_L and then $L \perp$ Mif and only if $[i_L, i_M] = 1$.

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